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Some subsets of the Hermitian curve

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Abstract

Three types of subsets of $PG(2, q^2)$ of type $(0, 1, 2, q + 1)$ are defined, namely $\mathcal{C}_F$-sets, K-sets and H-sets. These subsets may also be obtained as the intersection of the Hermitian curve $\mathcal{H}$ with Baer subpencils of lines with vertex not on $\mathcal{H}$. It is shown that under the Bruck–Bose representation these subsets correspond to three-dimensional elliptic quadrics, quadratic cones and hyperbolic quadrics, respectively contained in suitable hyperplanes of $PG(4, q)$.

1. Introduction

A *Hermitian curve* of the Desarguesian projective plane $PG(2, q^2)$ of square order q^2 is the set $\mathcal{H}$ of the absolute points of a unitary polarity. It has $q^3 + 1$ points; every line of $PG(2, q^2)$ meets $\mathcal{H}$ either in a Baer subline (such lines are called *secant*) or exactly in one point (such lines are called *tangent*). Through each point of $\mathcal{H}$ there pass q^2 secant lines and one tangent line and through each point not on $\mathcal{H}$ there pass $q + 1$ tangent lines, that form a *Baer subpencil* (i.e. a Baer subline in the dual plane), and $q^2 - q$ secant lines. In the following we will make use of the Bruck and Bose representation of $PG(2, q^2)$ in $PG(4, q)$. Let ℓ_∞ be a line of $PG(2, q^2)$, that we consider as the line at infinity. Let Σ_∞ be a hyperplane of $PG(4, q)$ and let $\mathcal{S}$ be a regular spread of Σ_∞. The points of the affine plane $PG(2, q^2)\backslash\ell_\infty$ are represented by the points of $PG(4, q)\backslash\Sigma_\infty$; the lines of $PG(2, q^2)\backslash\ell_\infty$ are represented by the planes of $PG(4, q)$ not contained in Σ_∞ and which meet Σ_∞ in a line of $\mathcal{S}$. The incidence relation of $PG(2, q^2)\backslash\ell_\infty$ is represented by set theoretic inclusion in $PG(4, q)$. We complete this representation by the addition of the points of ℓ_∞ that are represented in $PG(4, q)$ by the lines of the spread $\mathcal{S}$.

A Baer subplane containing $q+1$ points on ℓ_∞ is represented by a *transversal plane*, i.e. a plane of $PG(4, q)$ not contained in Σ_∞ and containing no elements of $\mathcal{S}$, and vice versa.

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A Baer subplane meeting ℓ_∞ in a unique point is represented by the ruled cubic ν_2^3, also called the *twisted ladder* [6, 10]. A Baer subline meeting ℓ_∞ in a point is represented by a line not contained in Σ_∞, and vice versa. A Baer subline with no point on ℓ_∞, contained in a line ℓ of $PG(2, q^2)$, is represented by a non-singular conic contained in the plane which represents ℓ. A conic of $PG(4, q)$, not contained in Σ_∞, which represents a Baer subline of $PG(2, q^2)$, is called a *Baer conic*. A Baer subline contained in ℓ_∞ is represented by a regulus contained in $\mathcal{S}$, and vice versa. A non-singular conic contained in Σ_∞ meeting the lines of $\mathcal{S}$ in points that belong to lines of a regulus is called an *R-conic*. From now on if P is a point of $PG(2, q^2)$, the corresponding point or line of $PG(4, q)$ will be denoted by P^*. The same notation will be used for subsets of $PG(2, q^2)$.

If ℓ_∞ meets the Hermitian curve $\mathcal{H}$ in $q + 1$ points, then the corresponding set $\mathcal{H}^*$ is a non-singular quadric of $PG(4, q)$ that meets the lines of $\mathcal{S}$ in lines of a regulus. For more details about the Bruck–Bose representation see [4, 5, 2, 6].

In this paper we study three types of subsets of $PG(2, q^2)$ of type $(0, 1, 2, q + 1)$ with respect to lines. The first ones, namely the $\mathcal{C}_F$-sets, are defined in [8] as the set of points of intersection of corresponding lines under a collineation induced by a suitable semilinear map between two pencils of lines of $PG(2, q^2)$ (see Section 3). Let $\mathcal{P}$ and $\mathcal{P}'$ be two Baer subpencils with vertices C and V, respectively. If the line $C \vee V$ belongs to $\mathcal{P}$ and not to $\mathcal{P}'$, then the set of points of intersection between the lines of $\mathcal{P}$ and the lines of $\mathcal{P}'$ defines a K-set. If the line $C \vee V$ does not belong to $\mathcal{P}$ or $\mathcal{P}'$, then the set of points of intersection between the lines of $\mathcal{P}$ and the lines of $\mathcal{P}'$ defines an H-set. In Sections 3–5 we prove that under the Bruck–Bose representation the $\mathcal{C}_F$-sets, K-sets and H-sets correspond to three-dimensional elliptic quadrics, quadratic cones and hyperbolic quadrics, respectively. Moreover each of these quadrics intersect Σ_∞ in an R-conic. Conversely, we prove that every three-dimensional elliptic quadric, quadratic cone and hyperbolic quadric meeting Σ_∞ in an R-conic is the Bruck–Bose representation of a $\mathcal{C}_F$-set, a K-set and an H-set, respectively. Finally we prove that these subsets can be obtained as the intersection of a Hermitian curve $\mathcal{H}$ with Baer subpencils of lines with vertex not on $\mathcal{H}$.

2. Representation of Baer subpencils

Lemma 2.1. *Every Baer subpencil of lines of $PG(2, q^2)$ with vertex C containing ℓ_∞ is represented by a hyperplane H of $PG(4, q)$ different from Σ_∞ such that $H \cap \Sigma_\infty$ contains the unique spread line C^*. Conversely every hyperplane of $PG(4, q)$ different from Σ_∞ represents a Baer subpencil containing ℓ_∞.*

Proof. Let $\mathcal{P}$ be a Baer subpencil of lines of $PG(2, q^2)$ containing ℓ_∞, with vertex C and let m be a line not through C. Then $m \cap \mathcal{P}$ is a Baer subline m_o which corresponds in $PG(4, q)$ to a line m_o^* not contained in Σ_∞. Hence $\mathcal{P}$ corresponds to the hyperplane (different from Σ_∞) spanned by the lines m_o^* and C^*. Conversely let $\mathcal{J}$ be a hyperplane of $PG(4, q)$ different from Σ_∞. There exists a unique line C^* of $\mathcal{S}$ contained in the plane $\mathcal{J} \cap \Sigma_\infty$. A line contained in $\mathcal{J}$, skew with C^*, represents a Baer subline m_o contained in a line not through C, so $\mathcal{J}$ represents the Baer subpencil $\{C \vee P : P \in m_o\}$. □

Lemma 2.2. *Every Baer subpencil of lines of $PG(2, q^2)$ with vertex not on ℓ_∞ is represented by a cone of $PG(4, q)$ with vertex a point not on Σ_∞ which projects a regulus contained in the spread $\mathcal{S}$, and vice versa.*

Proof. Let $\mathcal{P}$ be a Baer subpencil of lines of $PG(2, q^2)$ with vertex a point C not on ℓ_∞. Then $\mathcal{P}$ intersects ℓ_∞ in a Baer subline ℓ_o which corresponds to a regulus $\mathcal{R}$ of Σ_∞. Therefore $\mathcal{P}$ corresponds to the cone of $PG(4, q)$ with vertex C^* which projects the lines of $\mathcal{R}$. Conversely let Γ be a cone with vertex a point $C^* \notin \Sigma_\infty$ which projects the lines of a regulus $\mathcal{R}$ contained in $\mathcal{S}$. The regulus $\mathcal{R}$ represents a Baer subline ℓ_o of ℓ_∞; hence Γ represents the Baer subpencil formed by the lines through C which intersect ℓ_∞ in the subline ℓ_o. $\square$

Lemma 2.3. *Every Baer subpencil of lines of $PG(2, q^2)$, not containing ℓ_∞, with vertex on ℓ_∞ is represented by a cone of $PG(4, q)$ with vertex a line of $\mathcal{S}$ which projects a Baer conic contained in $PG(4, q)\backslash\Sigma_\infty$, and vice versa.*

Proof. Let $\mathcal{P}$ be a Baer subpencil of lines of $PG(2, q^2)$, not containing ℓ_∞, with vertex a point C on ℓ_∞. Let ℓ be a line of $PG(2, q^2)$ not through C. The line ℓ intersects $\mathcal{P}$ in a Baer subline ℓ_o, with no points on ℓ_∞, which is represented by a Baer conic $\mathcal{C}$ contained in $PG(4, q)\backslash\Sigma_\infty$. Then $\mathcal{P}$ corresponds to the cone of $PG(4, q)$ with vertex the line C^* of $\mathcal{S}$ which projects the conic $\mathcal{C}$. Conversely let Γ be a cone with vertex a line $C^* \in \mathcal{S}$ which projects a Baer conic $\mathcal{C}$ contained in $PG(4, q)\backslash\Sigma_\infty$. The cone Γ represents the Baer subpencil formed by the lines joining C with the points of the Baer subline corresponding to the conic $\mathcal{C}$. $\square$

3. $\mathcal{C}_F$-sets

Let α_F be the involutory automorphism of the Galois field $GF(q^2)$ given by $\alpha_F : x \in GF(q^2) \longrightarrow x^q \in GF(q^2)$. Let A and B be two distinct points of $PG(2, q^2)$ and let $\mathcal{F}_A$ and $\mathcal{F}_B$ be the pencils of lines with vertices A and B, respectively. Denote by Φ an α_F-collineation between $\mathcal{F}_A$ and $\mathcal{F}_B$ which do not map the line $A \vee B$ onto the line $B \vee A$. A $\mathcal{C}_F$*-set* is the set of points of intersections of corresponding lines under Φ. In [8] it is proved that every such set is projectively equivalent to the algebraic curve with equation $x_1 x_2^q - x_3^{q+1} = 0$. Every $\mathcal{C}_F$-set has $q^2 + 1$ points; it is a set of type $(0, 1, 2, q + 1)$ with respect to lines of $PG(2, q^2)$ and every $(q+1)$-secant line meets a $\mathcal{C}_F$-set in a Baer subline. The $(q + 1)$-secant lines number in total $q - 1$ and all contain a common point C not on the $\mathcal{C}_F$-set. Those lines, together with the lines $C \vee A$ and $C \vee B$, form a Baer subpencil.

Proposition 3.1. *Let $\mathcal{O}$ be a $\mathcal{C}_F$-set and let ℓ_∞ be a $(q+1)$-secant line. Then in the Bruck–Bose representation with ℓ_∞ as the line at infinity, $\mathcal{O}^*$ is an elliptic quadric contained in a hyperplane H of $PG(4, q)$ meeting Σ_∞ in an R-conic.*

Proof. Since $\mathcal{O}$ is contained in a Baer subpencil with vertex C, by Lemma 2.1 $\mathcal{O}^*$ is a $(q^2 + 1)$-set contained in a hyperplane H of $PG(4, q)$ different from Σ_∞. We first prove that $\mathcal{O}^*$ is a $(q^2 + 1)$-cap of H. Let ℓ be a line of H different from C^*. The only Baer sublines contained in $\mathcal{O}$, different from $\ell_\infty \cap \mathcal{O}$, are represented by Baer conics. Therefore ℓ represents a Baer subline of $PG(2, q^2)$ not contained in $\mathcal{O}$. Since every Baer subline not contained in $\mathcal{O}$ meets $\mathcal{O}$ in not more than two points, ℓ intersects $\mathcal{O}^*$ in at most two points. For q odd, Barlotti's result [1] shows that $\mathcal{O}^*$ is an elliptic quadric of H. For q even, since $\mathcal{O}^*$ contains a non-singular conic, Brown's result [3] shows that $\mathcal{O}^*$ is again an elliptic quadric. Finally, since $\mathcal{O} \cap \ell_\infty$ is a Baer subline, $\mathcal{O}^*$ meets Σ_∞ in points that belong to a regulus contained in $\mathcal{S}$. □

Lemma 3.2. *Let $\ell = PG(1, K)$ be a projective line over a field K and let σ be an automorphism of K. For every subline ℓ' of the line ℓ, coordinatized over the subfield* $\mathrm{Fix}\,(\sigma) = \{x \in K : x^\sigma = x\}$*, there exists only one σ-collineation of the line ℓ into itself fixing pointwise the subline ℓ'.*

Proof. Let f, g be two σ-collineations of the line ℓ fixing pointwise the subline ℓ'. Then $f^{-1} \circ g$ is a projectivity of the line ℓ fixing pointwise ℓ'. Since $|\ell'| \geq 3$, from the fundamental theorem of projective geometry we have that $f^{-1} \circ g$ is the identity, so $f = g$. Now take three distinct points A, B, C on ℓ'. We may suppose that $A = \langle(0, 1)\rangle$, $B = \langle(1, 0)\rangle$, $C = \langle(1, 1)\rangle$; hence the σ-collineation induced by the σ-semilinear map

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1^\sigma \\ x_2^\sigma \end{pmatrix}$$

fixes pointwise ℓ'. □

Proposition 3.3. *Let ℓ_0 be a Baer subline of a line ℓ of $PG(2, q^2)$ and let A, B be two points not on ℓ such that the point $(A \vee B) \cap \ell$ is not on ℓ_0. Then there exists only one $\mathcal{C}_F$-set that meets ℓ in the Baer subline ℓ_0 which is generated by an α_F-collineation between the pencils with vertices A and B.*

Proof. Observe that there is a bijective map Φ between the set of the α_F-collineations of the line ℓ into itself and the set of the α_F-collineations between the pencils of lines $\mathcal{P}(A)$ and $\mathcal{P}(B)$ with vertices A and B. To every α_F-collineation f, Φ corresponds to the α_F-collineation Φ_f defined by

$$\Phi_f : r \in \mathcal{P}(A) \longrightarrow f(r \cap \ell) \vee B \in \mathcal{P}(B).$$

By the previous lemma there exists only one α_F-collineation f_ℓ of the line ℓ into itself fixing pointwise the Baer subline ℓ_0. Since f_ℓ does not fix $(A \vee B) \cap \ell$, Φ_{f_ℓ} does not map the line $A \vee B$ onto the line $B \vee A$. Moreover Φ_{f_ℓ} is the unique α_F-collineation between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ such that every point on ℓ_0 is a point of intersection of corresponding lines. Hence the $\mathcal{C}_F$-set defined by Φ_{f_ℓ} is the only one which is generated by an α_F-collineation between the pencils $\mathcal{P}(A)$ and $\mathcal{P}(B)$ that meets ℓ exactly in the points of the subline ℓ_0. □

Proposition 3.4. *Every elliptic quadric contained in a hyperplane of $PG(4, q)$, meeting Σ_∞ in an R-conic, represents a $\mathcal{C}_F$-set.*

Proof. Let $Q^-(3,q)$ be an elliptic quadric contained in a hyperplane H of $PG(4,q)$ which meets the lines of $\mathcal{S}$ in points that belong to lines of a regulus $\mathcal{R}$. Denote by C^* the unique line of the spread $\mathcal{S}$ contained in the plane $H \cap \Sigma_\infty$ and by A^*, B^* the points of intersection of $Q^-(3,q)$ and the two tangent planes to $Q^-(3,q)$ through C^*. The quadric $Q^-(3,q)$ represents a (q^2+1)-set $\mathcal{C}$ of $PG(2,q^2)$ contained in a Baer subpencil with vertex C. Let ℓ_0 be the Baer subline of ℓ_∞ corresponding to the regulus $\mathcal{R}$ and let $\mathcal{C}'$ be the unique $\mathcal{C}_F$-set containing the Baer subline ℓ_0 which is generated by an α_F-collineation between the pencils with vertices A and B. By Proposition 3.1 $\mathcal{C}'^*$ is an elliptic quadric contained in H. The quadrics $\mathcal{C}'^*$ and $\mathcal{C}^*$ both contain the non-singular conic ℓ_0^* and have at A^* and B^* the same tangent planes. This gives nine independent linear conditions satisfied by the equations of both quadrics. Hence $\mathcal{C}'^* = \mathcal{C}^*$. $\square$

Recall that if ℓ_∞ is a secant line to $\mathcal{H}$, then $\mathcal{H}^*$ is a non-singular quadric of $PG(4,q)$ that meets the lines of the spread in lines of a regulus. It is easy now to show that the Hermitian curve $\mathcal{H}$ always contains $\mathcal{C}_F$-sets.

Corollary 3.5. *Let H be a hyperplane of $PG(4,q)$ meeting $\mathcal{H}^*$ in an elliptic quadric such that $H \cap \mathcal{H}^*$ intersects Σ_∞ in an R-conic. Then $H \cap \mathcal{H}^*$ represents a $\mathcal{C}_F$-set contained in the Hermitian curve $\mathcal{H}$.*

Proof. Let π be a plane meeting $\Sigma_\infty \cap \mathcal{H}^*$ in an R-conic. From properties of the polarity of the quadric $\mathcal{H}^*$ there are at least $(q-1)/2$ hyperplanes containing the R-conic $\pi \cap \Sigma_\infty \cap \mathcal{H}^*$ and meeting the quadric $\mathcal{H}^*$ in an elliptic quadric. Let H be one such hyperplane; from the previous proposition $H \cap \mathcal{H}^*$ represents a $\mathcal{C}_F$-set contained in the Hermitian curve $\mathcal{H}$. $\square$

Let C be a point not on $\mathcal{H}$, let ℓ_∞ be a $(q+1)$-secant line through C, that we consider as the line at infinity in the Bruck–Bose representation, and let ℓ_A, ℓ_B be two tangent lines through C meeting $\mathcal{H}$ in A and B respectively.

Proposition 3.6. *The intersection of the Hermitian curve $\mathcal{H}$ with the Baer subpencil containing the lines $\ell_\infty, \ell_A, \ell_B$ is the only $\mathcal{C}_F$-set which is generated by an α_F-collineation between the pencils of lines with vertices A and B containing the Baer subline $\mathcal{H} \cap \ell_\infty$.*

Proof. Let $\mathcal{P}$ be the Baer subpencil containing ℓ_∞, ℓ_A, ℓ_B and let C be the vertex of $\mathcal{P}$. Observe that every line of $\mathcal{P}$ different from ℓ_A and ℓ_B is a $(q+1)$-secant line with respect to the Hermitian curve. Indeed if $\mathcal{P}$ contained a third tangent line, then it would be coincident with the Baer subpencil formed by the tangent lines through C, which is not possible. Hence $|\mathcal{H} \cap \mathcal{P}| = q^2+1$ and $\mathcal{H} \cap \mathcal{P}$ is represented by an elliptic quadric of the hyperplane $\mathcal{P}^*$ which meets the lines of $\mathcal{S}$ in lines of a regulus. Indeed $\mathcal{H}^*$ is the non-singular quadric of $PG(4,q)$, $\mathcal{P}^*$ is a hyperplane of $PG(4,q)$ and by Lemma 2.1 $\mathcal{P}^* \cap \Sigma_\infty$ contains the unique spread line C^*; hence $\mathcal{P}^* \cap \Sigma_\infty \cap \mathcal{H}^*$ is an R-conic. It follows that $|\mathcal{H}^* \cap \mathcal{P}^*| = q^2+1$; then $\mathcal{H}^* \cap \mathcal{P}^*$ is an elliptic quadric. Hence, by Corollary 3.5, $\mathcal{H}^* \cap \mathcal{P}^*$ represents a $\mathcal{C}_F$-set $\mathcal{O}$ contained in the Hermitian curve. Observe that the points A^* and B^* are the points of intersection of $\mathcal{H}^* \cap \mathcal{P}^*$ and the two tangent planes to $\mathcal{H}^* \cap \mathcal{P}^*$ through the line C^*. So, as in the proof of Proposition 3.4, $\mathcal{O}$ is the unique $\mathcal{C}_F$-set containing the

Baer subline $\mathcal{H} \cap \ell_\infty$ which is generated by an α_F-collineation between the pencils of lines with vertices A and B. $\square$

4. K-sets

Let $\mathcal{P}$ and $\mathcal{P}'$ be two Baer subpencils of $PG(2, q^2)$ with vertices C and V respectively such that the line $C \vee V$ belongs to $\mathcal{P}$ and not to $\mathcal{P}'$. A *K-set* $\mathcal{C}$ is the set of $q^2 + q + 1$ points of intersection between the lines of $\mathcal{P}$ and the lines of $\mathcal{P}'$. In [7] it is proved that every such set is of type $(0, 1, 2, q + 1)$ with respect to lines of $PG(2, q^2)$ and it is easy to see that every $(q + 1)$-secant line is a Baer subline. The $(q + 1)$-secant lines number in total $2q + 1$ and all contain either C or V.

Proposition 4.1. *Let $\mathcal{C}$ be a K-set and let ℓ_∞ be a $(q+1)$-secant line containing C. Then in the Bruck–Bose representation with ℓ_∞ as the line at infinity $\mathcal{C}^*$ is a quadratic cone contained in a hyperplane of $PG(4, q)$ meeting Σ_∞ in an R-conic.*

Proof. By Lemmas 2.1 and 2.2 and the fact that $C \notin \mathcal{P}'$, $\mathcal{C}$ is represented by the set $\mathcal{C}^*$ consisting of q^2+q+1 points obtained as the intersection of the hyperplane $H = \mathcal{P}^*$ with the cone $\mathcal{P}'^*$ with vertex $V^* \in H$ projecting the regulus corresponding to the Baer subline $\mathcal{C} \cap \ell_\infty$. Hence $\mathcal{C}^*$ is the cone of H with vertex V^* projecting the conic $H \cap \Sigma_\infty \cap \mathcal{P}'^*$, whose points belong to lines of a regulus contained in $\mathcal{S}$. $\square$

Proposition 4.2. *Every quadratic cone γ with vertex $V \notin \Sigma_\infty$, contained in a hyperplane H of $PG(4, q)$ and projecting an R-conic σ, represents a K-set.*

Proof. The conic σ meets the lines of $\mathcal{S}$ in lines of a regulus $\mathcal{R}$. Since γ is the intersection of the cone Γ with vertex V projecting the regulus $\mathcal{R}$ and the hyperplane H, by Lemma 2.1 γ represents a K-set. $\square$

In order to show that a Hermitian curve always contains K-sets, take a point C not on $\mathcal{H}$ and a Baer subpencil $\mathcal{P}$ with vertex C containing a unique tangent line ℓ_V meeting $\mathcal{H}$ in V.

Proposition 4.3. *Let $\ell_\infty \in \mathcal{P}$ be a $(q + 1)$-secant line to $\mathcal{H}$. Then the intersection of the Hermitian curve $\mathcal{H}$ with the Baer subpencil $\mathcal{P}$ is a K-set.*

Proof. The Hermitian curve $\mathcal{H}$ is represented by a parabolic quadric $Q(4, q)$. Let H be the hyperplane of $PG(4, q)$ representing the Baer subpencil $\mathcal{P}$. Since $|\mathcal{P} \cap \mathcal{H}| = q^2+q+1$ and $C \notin \mathcal{H}$, $H \cap Q(4, q)$ is a cone with vertex V^* projecting the conic $\Sigma_\infty \cap H \cap Q(4, q)$. From Proposition 4.2, $\mathcal{H} \cap \mathcal{P}$ is a K-set. $\square$

5. H-sets

Let $\mathcal{P}$ and $\mathcal{P}'$ be two Baer subpencils of $PG(2, q^2)$ with vertices C and V respectively such that the line $C \vee V$ does not belong to $\mathcal{P}$ or $\mathcal{P}'$. An *H-set* is the set of $(q + 1)^2$ points of intersection of the lines of $\mathcal{P}$ and the lines of $\mathcal{P}'$. It is easy to see that every such set is of type $(0, 1, 2, q+1)$ with respect to lines of $PG(2, q^2)$ and that every $(q+1)$-secant line is a Baer subline.

Proposition 5.1. *Let $\mathcal{C}$ be an H-set and ℓ_∞ be a $(q+1)$-secant line through C. Then in the Bruck–Bose representation with ℓ_∞ as the line at infinity $\mathcal{C}^*$ is a hyperbolic quadric contained in a hyperplane of $PG(4, q)$ meeting Σ_∞ in an R-conic.*

Proof. By Lemmas 2.1 and 2.2, $\mathcal{C}^*$ is obtained as the intersection of the hyperplane $H = \mathcal{P}^*$ and the cone $\mathcal{P}'^*$ with vertex V^* projecting the lines of the regulus $\mathcal{R}$ corresponding to the Baer subline $\ell_\infty \cap \mathcal{C}$. Since $V^* \notin H$, H intersects the cone $\mathcal{P}'^*$ in a hyperbolic quadric. Moreover, since $C \notin \mathcal{C}$, $H \cap \Sigma_\infty \cap \mathcal{C}^*$ is an R-conic which meets the lines of $\mathcal{S}$ in the lines of the regulus $\mathcal{R}$. □

An H-set $\mathcal{C}$ contains exactly $3(q+1)$ Baer sublines. Recall that a Baer subline corresponds either to a line of $PG(4, q)$ not contained in Σ_∞ or to a non-singular conic contained in a plane of $PG(4, q)$ meeting Σ_∞ in a line of the spread $\mathcal{S}$. Hence the only Baer sublines contained in $\mathcal{C}$ correspond in $PG(4, q)$ to the $2(q+1)$ lines of the two reguli of $\mathcal{C}^*$ and to the $q+1$ non-singular conics of the linear flock of $\mathcal{C}^*$ defined by the line C^*.

Proposition 5.2. *Every hyperbolic quadric contained in a hyperplane of $PG(4, q)$, meeting Σ_∞ in an R-conic, represents an H-set.*

Proof. Let $Q^+(3, q)$ be a hyperbolic quadric contained in a hyperplane H of $PG(4, q)$ which meets Σ_∞ in lines of a regulus $\mathcal{R}$. By Lemma 2.1 the hyperplane H represents a Baer subpencil $\mathcal{P}$ with vertex the point C such that C^* is the unique line of the spread $\mathcal{S}$ contained in the plane $H \cap \Sigma_\infty$. Let r_0^*, s_0^* be two lines of one regulus of $Q^+(3, q)$, that represent two Baer sublines of $PG(2, q^2)$ contained in two lines r and s, respectively. Let V be the point of intersection of r and s and let $\mathcal{P}'$ be the Baer subpencil formed by the lines joining V with the points of the Baer subline ℓ_0 of ℓ_∞ which corresponds to the regulus $\mathcal{R}$. Observe that $V = r \cap s$ lies on no line of $\mathcal{P}$ and also C does not lie on ℓ_0. Thus the line $C \vee V$ does not belong to $\mathcal{P}$ or $\mathcal{P}'$. Hence the points of intersection of the lines of $\mathcal{P}$ and the lines of $\mathcal{P}'$ form an H-set $\mathcal{C}$. By the previous proposition $\mathcal{C}^*$ is a hyperbolic quadric contained in the hyperplane H. Observe that the quadrics $Q^+(3, q)$ and $\mathcal{C}^*$ have in common the conic section $Q^+(3, q) \cap \Sigma_\infty$ and the two skew lines r_0^*, s_0^*, so they have to coincide ([9], Theorem II, p. 114). □

Finally, let ℓ_0 be a Baer subline of a line ℓ of $PG(2, q^2)$ and let V and C be two points not on ℓ such that the point $(C \vee V) \cap \ell$ is not on ℓ_0. Then there exists only one H-set $\mathcal{C}$ obtained as the intersection of Baer subpencils with vertices C and V containing the Baer subline ℓ_0. The set $\mathcal{C}$ is obviously obtained as the intersection of the Baer subpencil $\mathcal{P}$ with vertex C projecting the Baer subline ℓ_0 with the Baer subpencil $\mathcal{P}'$ with vertex V projecting ℓ_0.

In order to show that a Hermitian curve of $PG(2, q^2)$ always contains H-sets, take a point C not on $\mathcal{H}$ and a Baer subpencil $\mathcal{P}$ with vertex C containing all secant lines to $\mathcal{H}$.

Proposition 5.3. *Let $\ell_\infty \in \mathcal{P}$ be a $(q+1)$-secant line to $\mathcal{H}$. Then the intersection of the Hermitian curve $\mathcal{H}$ with the Baer subpencil $\mathcal{P}$ is an H-set.*

Proof. The Hermitian curve $\mathcal{H}$ is represented by a parabolic quadric $Q(4, q)$. Let H be the hyperplane of $PG(4, q)$ representing the Baer subpencil $\mathcal{P}$. Since $C \notin \mathcal{H}$ and $|\mathcal{P} \cap \mathcal{H}| = (q+1)^2$, $H \cap Q(4, q)$ is a hyperbolic quadric containing the conic $\Sigma_\infty \cap H \cap Q(4, q)$. From Proposition 5.2, $\mathcal{H} \cap \mathcal{P}$ is an H-set. □

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